MCS-033.

Block 2, Unit 1

Basic Properties of a graph.

Graph : A graph G, is a set of vertices & edges.
It is denoted as $G = (V, E)$, where V is the vertex set & E is edge set.

Undirected graph : An undirected graph G is a finite non-empty set V together with a set E consisting of pairs of points of V.
denoted as = $G = (V, E)$

$(u, v) \in G \quad \therefore (v, u) \in G$

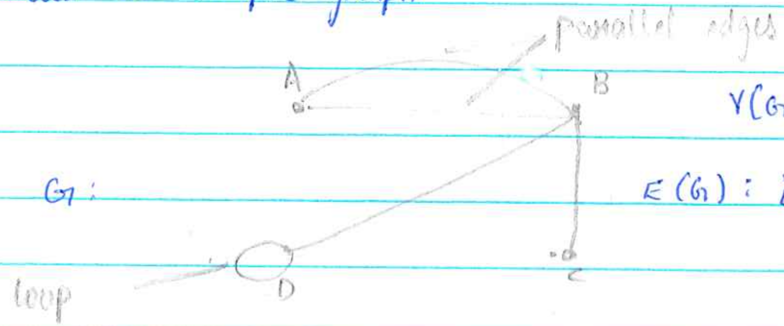
Directed graph / digraph : A directed graph G is a finite non-empty V together with subset E of the cartesian product set $V \times V$.

Loop : It's an edge that joins a vertex to itself.

Parallel edges / multiple edges : 2 or more edges joining the same vertices.

Multigraph : A graph with multiple edges is called multigraph.

Simple graph : An undirected graph with no parallel edges or loops is called a simple graph.



$V(G) = \{A, B, C, D\}$

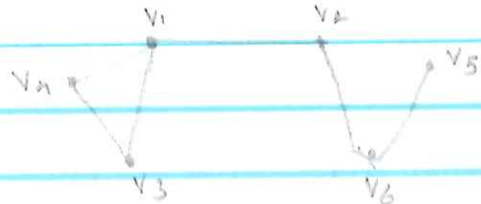
$E(G) = \{(AB, BD, BC)\}$.

Finite graph : A graph $G = \{V, E\}$ is a finite graph if the vertex set V is a finite set.

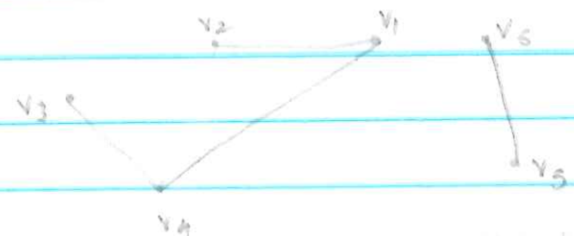
Infinite graph: A graph $G = \{V, E\}$ is an infinite graph if the vertex set V is an infinite set.

Null graph: It is a graph is no edges. It is denoted as N_5.

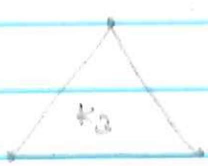
Connected graph: A graph is connected when there is path between every pair of vertices.

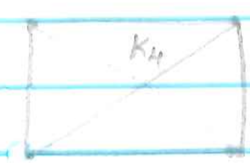


Disconnected graph: A graph is disconnected when there is no path between atleast one pair of vertices.



Complete graph: A graph G is said to be a complete graph, if each vertex is connected to every other vertex in a graph by an edge. It is denoted as K_n. n is the no. of vertices.





no. of vertices n, no. of edges $= \dfrac{n(n-1)}{2}$

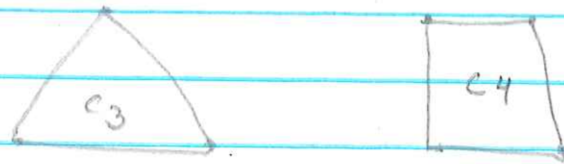
Q) How many edges do K_{10} have?

no. of edges $= \dfrac{n(n-1)}{2}$, where $n = 10$

$= \dfrac{10(10-1)}{2} = \underline{45}$

Star topology graph : A graph in which a no. of vertices are connected to a central vertex.

Cycles : A cycle C_n is a graph n vertices $\{x_1, \ldots x_n\}$ where $E(C_n) = \{x_i x_{i+1} : 1 \le i \le n-1\} \cup \{x_n x_1\}$.

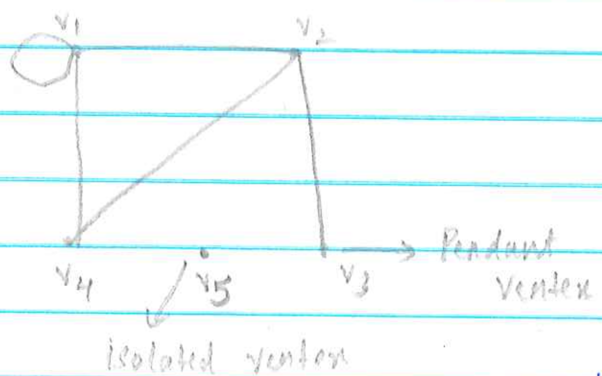


9) How many edges does C_{10} has ?

no. of edges = no. of vertices

$\therefore$ no. of edges in C_{10} = 10

Degree of a ~~graph~~ vertex : Degree of a vertex x is the no. of edges incident with x.



$d(v_1) = 4$ (loop is counted as 2 edges)

$d(v_2) = 2$

$d(v_4) = 2$

$d(v_3) = 1$

Isolated vertex : Is a vertex ~~is~~ with degree 0.

Pendant vertex : Is a vertex with degree 1.

$\delta(G)$: minimum vertex of graph $\delta(G) = 0$ (above graph)

$\Delta(G)$: maximum vertex of graph. $\Delta(G) = 4$ (" ")

Handshaking theorem:

$\boxed{\text{Sum of degrees} = 2 \times \text{no. of edges.}}$

Proof.

Consider the set $S = \{(x, e) : x \in V(G), e \in E(G), x \text{ is an endpoint of } e\}$

~~Con~~ choose a vertex v_i from $V(v_i \in V)$. This can be done in p ways.

$d_i = d(v_i)$. Therefore, there are d_i edges incident with v_i. These edges give d_i elements of S.

Adding over all the vertices of G,

we get, $|S| = \sum_{i=1}^{p} d_i$ — (1)

Now, choose an edge e in $E(G)$. This can be done in q ways. This edge has precisely 2 endpoints, this gives 2 elements of S. Adding over all edges,

we get $|S| = 2q$ — (2)

this is because each edge is counted twice, once for each vertex.

Equating (1) & (2)

$$\sum_{i=1}^{p} d_i = 2q.$$

Corollary 1: The sum of degree of all vertices of a graph is even.

Corollary 2: Any graph can only have an even number of odd vertices.

Proof: Let G be a (p, q) graph and let $\{x_1, \dots x_t\}$ be the set of its odd vertices and $\{x_{t+1}, \dots x_p\}$ be the set of even vertices.

Let $d_G(x_i) = 2c_i + 1$, $1 \le i \le t$ and $d_G(x_i) = 2x_i$ $t+1 \le i \le p$.

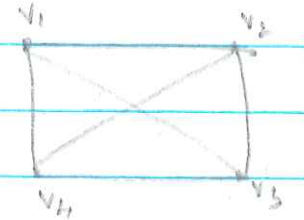
$$2q = \sum_{i=1}^{p} d_G(x_i)$$

$$\Rightarrow 2q = \sum_{i=1}^{t} 2c_i + 1 + \sum_{i=t+1}^{p} 2x_i$$

$$= 2(c_1 + c_2 + \cdots + c_t) + t + 2(x_{t+1} + \cdots + x_p)$$

which shows, that t is even.

Regular graph: A graph in which all the vertices are of equal degree. It is denoted as n-regular graph. (n is the degree).



$d(v_1) = 3$

$d(v_2) = 3$

$d(v_3) = 3$

$d(v_4) = 3$

Q) Find the no. of edges of a 4-regular graph with 6 vertices.

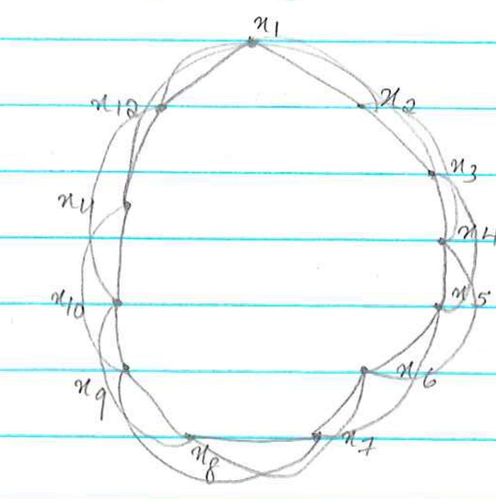
Sum of degree = 2 x no. of edges

$6 \times 4 = 2 \times$ no. of edges

$24/2 =$ no. of edges

$12 =$ no. of edges

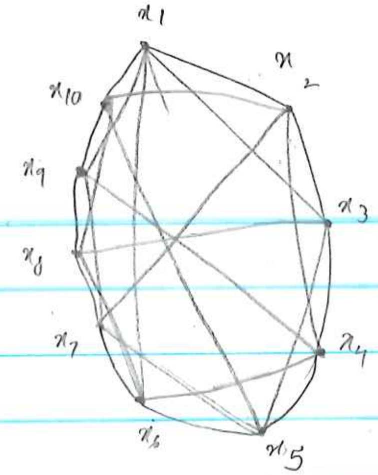
Q) Construct a 4-regular graph G with 12 vertices.



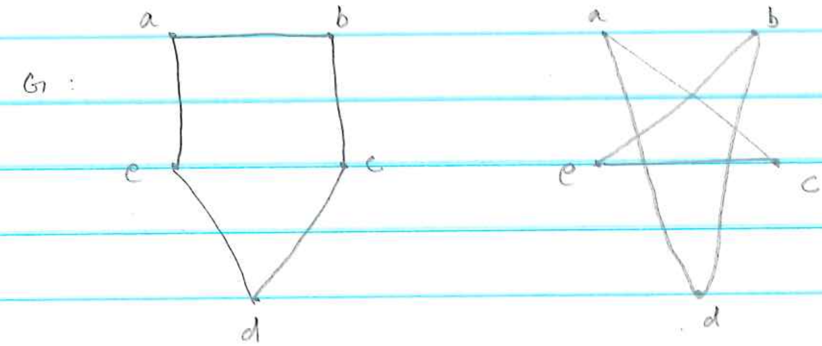
1. Join x_i to x_{i+1} —
2. Join x_n to x_1
3. Join x_i to x_{i+2} —
4. Join $\frac{P}{2} = 6$; x_i to x_{i+6}

Q) Construct a 5-regular graph on 10 vertices.

$\frac{P}{2} = 5 = \frac{10}{2}$



Complement of a graph : Let G be ~~a graph~~ (p, q) graph.
By ~~it~~. ~~A~~ Complement of a graph G is $\bar{G}$ with $V(\bar{G}) = V(G)$ & $E(\bar{G}) = \{xy : xy \notin E(G) \, , \, x, y \in V(G)\}$



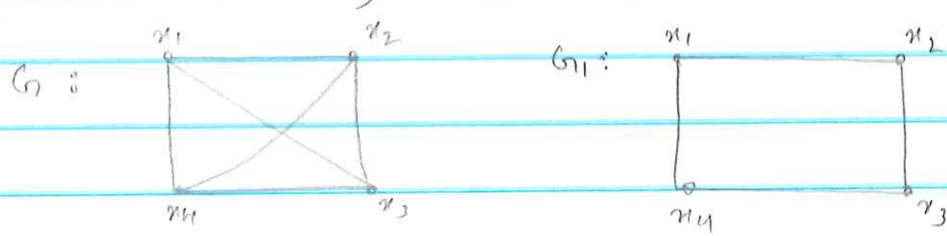
$$\boxed{\text{no. of edges in } \bar{G} = \frac{P(P-1)}{2} - q}$$

$P \rightarrow$ no. of vertices, $q \rightarrow$ no. of edges in G.

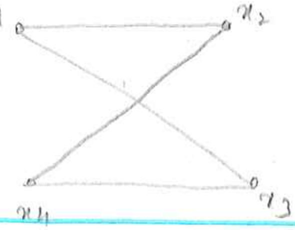
$(V(G), E(G))$

Subgraph : Let G be a graph (p, q) graph. A graph H is said to be a ~~subset~~ subgraph of G if $V(H) \subseteq V(G)$ & $E(H) \subseteq E(G)$.

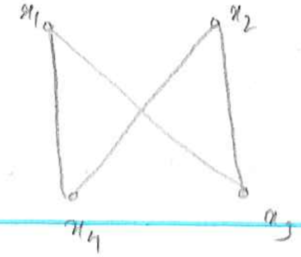
Spanning subgraph : If H is a subgraph of a graph G, such that $V(H) = V(G)$ & $E(H) \subseteq E(G)$, then ~~the~~ H is a spanning subgraph of G.
(vertex set is same).



G_2 :



G_3 :

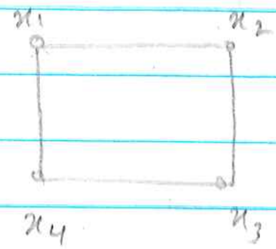


Q) Is every subgraph of a regular graph regular ?

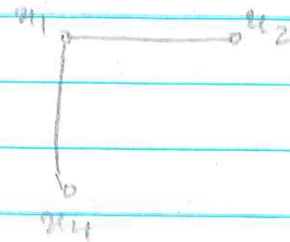
No.

for example, any cycle is regular. However if you remove one of its edges, you get a subgraph which is not regular.

C_4 :

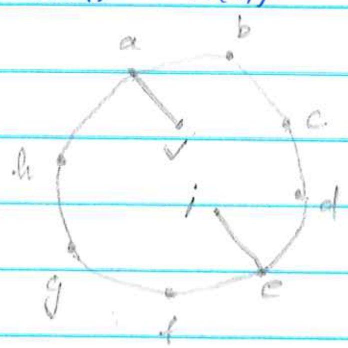


Subgraph 1 :

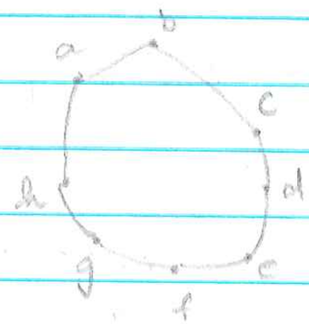


the above is not regular.

Q) Give an example of a subgraph H of a graph G with $\delta(G) < \delta(H)$ & $\Delta(H) < \Delta(G)$.



G



H.

$\delta(G) = 1$ and $\delta(H) = 2$

$\therefore \delta(H) < \delta(H)$

$\Delta(H) = 2$ and $\Delta(G) = 3$

$\therefore \Delta(H) < \Delta(G)$

Isomorphic graph: Let $G = (V(G), E(G))$, $H = (V(H), E(H))$ be two graphs. The two graphs are said to be isomorphic if there is a one to one correspondance (or one-one & onto) between their vertices & between their edges.

It is denoted as $G_1 = G_2$ / $G_1 \cong G_2$.

Conditions for isomorphic graph:

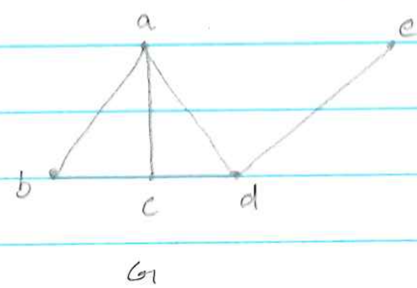
↳ same no. of vertices

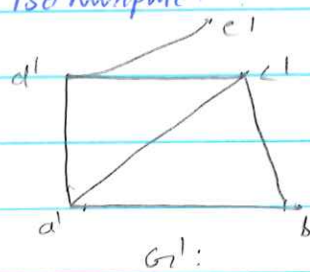
↳ same no. of ~~vertices~~ edges

↳ equal no. of vertices with same degree

↳ same degree sequence & same cycle vector.
↳ (cycle formation).

Q) S.T. the following graphs are isomorphic.





Condition 1: Same no. of vertices

in $G = 5$ in $G' = 5$

Condition 2: Same no. of edges

in $G = 6$ in $G' = 6$

Condition 3:

equal no. of vertices with same degree.

G: $d(a) = 3$ $d(b) = 2$ $d(c) = 3$ $d(d) = 3$ $d(e) = 1$

G': $d(a') = 3$ $d(b') = 2$ $d(c') = 3$ $d(d') = 3$ $d(e') = 1$

Correspondence relation:

a – a' e – e'

b – b'

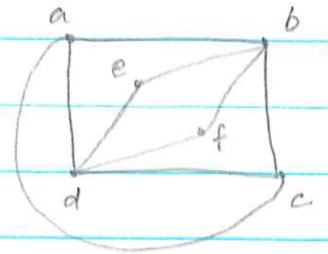
c – c'

d – d'

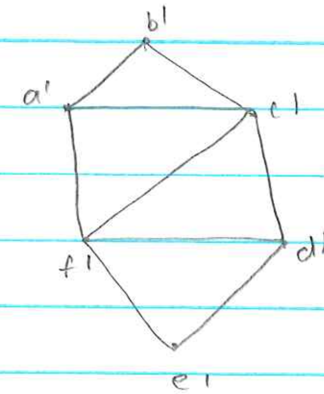
condition 4: have same degree sequence & same cycle vector.
∴ G & G' are isomorphic.

Q) Are the 2-graphs isomorphic?

G:



G':



G: d(a) = 3 d(b) = 4 d(c) = 3 d(d) = 4 d(e) = 2 d(f) = 2
G': d(a') = 3 d(b') = 2 d(c') = 4 d(d') = 3 d(e') = 2 d(f') = 4

Same no. of vertices = 6
Same no. of edges = 9
correspondance
vertices of degree 4 : b – c' | f'
d – f' | c'

vertices of degree 3 : a – a'
c – d'

∵ adjacency relationship is violated in vertices having degree 3 & 4.
∴ G & G' are not isomorphic.

MCS-033.

block 2 Unit-2.

Connectedness.

Walk: A walk in a graph G is a finite sequence $W = \{v_0, e_1, v_1, \ldots e_k, v_k\}$, where $v_0, v_1, v_2, \cdots v_k$ are vertices of G and $e_1, e_2 \cdots e_k$ are edges ~~of e~~ joining the vertices v_{i-1} & v_i, $1 \leq i \leq k$.

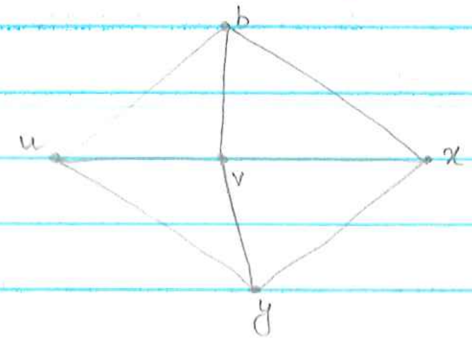
W is a walk from v_0 to v_k, or W is a $v_0 - v_k$ walk or W is a walk joining v_0 & v_k.

v_0 is called the initial vertex & v_k is called the end vertex.

No. of edges in the walk W is called the length of the walk ($l(W)$).

In a walk, vertices & edges can be repeated.

Example:



$W = \{u, ub, b, bx, x, xy, y, yu, u, ub, b, bx, x, xv, v\}$

$l(W) = 7$

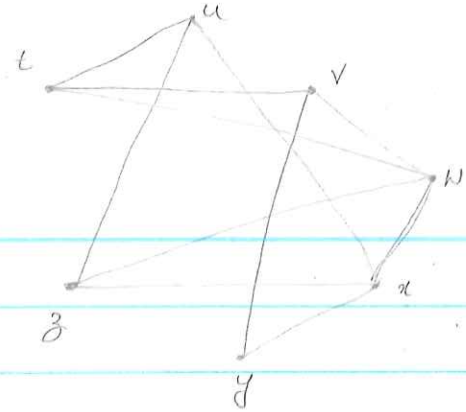
Path: A walk W is called a path if all the vertices & edges are distinct.

closed walk: A u-v walk is said to be closed if $u = v$.

open walk: A u-v walk is said to be open if $u \neq v$.

cycle: A walk in which all the edges are distinct and only repeated vertex is the first vertex, this being the same as the last vertex, is called a cycle.

E3)



i) a u-v walk that is not a path,

~~W = {t, tu, u, uz, z, zw, w, wt, t, tv, v}~~

~~W = {t, tu, u, uz, z, zw, w, wx, v, vt, t, tu, u}~~

W = {u, ut, t, tw, wx, xz, zw, w, wt, t, tv, v}

ii) a (u-u) walk that is not a cycle.

W = {u, ut, t, tw, w, wv, v, vt, t, tu, u}

iii) a (u-u) cycle of minimum length.

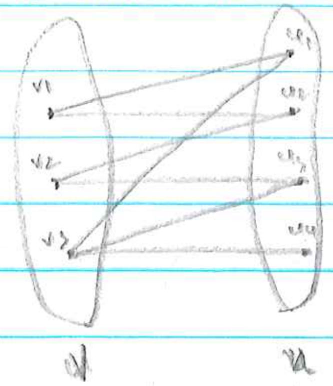
W = {u, uz, z, zx, x, xu, u}

$l(W) = 3$

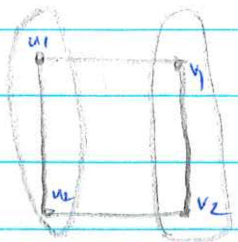
If W is a u-v walk joining two distinct vertices u & v, then there is a path joining u & v contained in the walk.

Bipartite graph / bigraph :

A bipartite graph is a graph whose vertices can be divided into two disjoint & independent sets U & V such that every edge connects a vertex in U ~~&~~ to one in V.



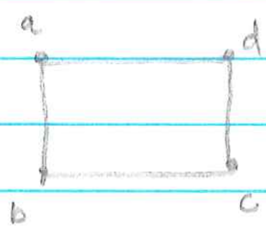
Q)

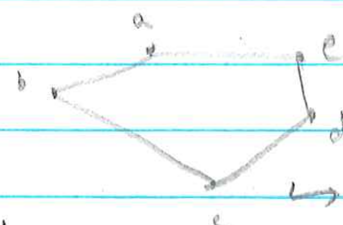


→ not bipartite.

b/coz, u_1 & u_2 are connected

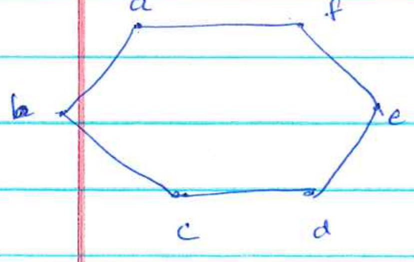
||y v_1 & v_2 are connected.



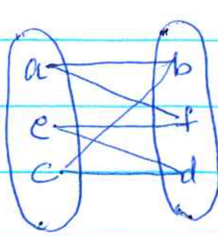


↳ not bipartite.

a b
c e
d

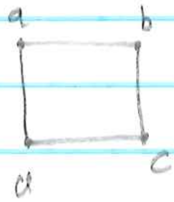


d can't include coz de is connected



Theorem : A graph G is bipartite if and only if G does not contain any cycle of odd length as a subgraph.

$\therefore$ C_n is not bipartite whenever n is odd.



C_5 is not bipartite : length of cycle in C_5 is 5.

(X) Q) Draw a graph (connected) which can be both regular & bipartite?

We know, that a graph is bipartite if it does not contain any odd length cycle.

Every cycle graph is regular.

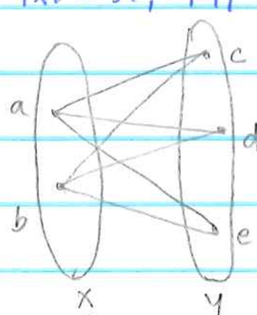
So, C_n is bipartite if n is even.

e.g. C_4, C_6, C_8, C_{10} .....

Complete bipartite graph :

A complete bipartite graph is a bipartite graph $G(X, Y)$ in which each $x \in X$ is joined to every $y \in Y$ ie. G is also a comple graph.

denoted by : $K_{r,s}$. $|X| = r$, $|Y| = s$



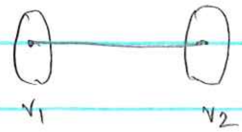
$K_{2,3}$

$r = 2$, $s = 3$

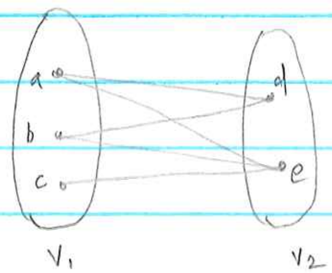
Complete bipartite graphs are not complete graphs.

Complete bipartite graph which is a complete graph (only one condition):

$m = n = 1$



Q) How many vertices and edges $K_{m,n}$ has?



Note : m is no. of vertices in V_1

n is no. of vertices in V_2.

no. of vertices $= m + n$

$= 3 + 2$

$= \underline{\underline{5}}$

no. of edges $= m * n$

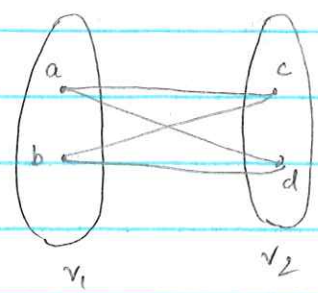
$= 3 \times 2$

$= \underline{\underline{6}}$

a) When $K_{m,n}$ is regular?

A complete bipartite graph is regular if $m = n$.

eg:

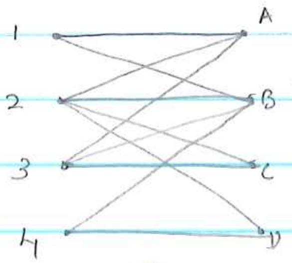


G

G is regular.

Matching in a bipartite graphs.

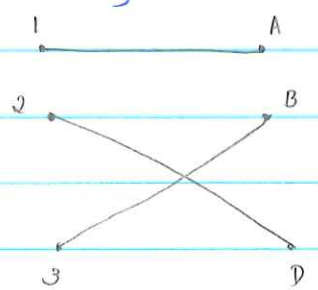
A matching in a bipartite graph G is a set of edges such that no two edges have a common end vertex.

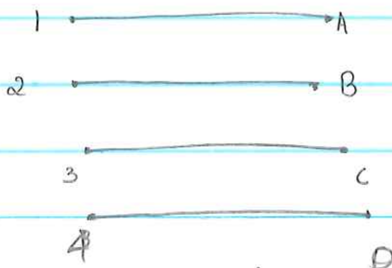


G.

find a matching for this bipartite graph.

Matching in G.





complete matching.

complete matching :

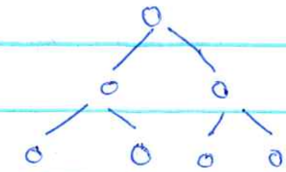
A matching of X into Y is called a complete matching of X & Y if there is an edge incident with every vertex in X.

Let $G = G(X, Y)$ be a bipartite graph. A complete matching of X into Y exists in G if and only if $|A| \leq |R(A)|$ for every subset A of X, where $|A|$ denotes the number of elements in A. and $R(A)$ denotes the set of vertices in Y that are adjacent to the vertices in A.

Tree : A tree is a connected graph with no cycles.

Forest : A forest is a graph, whose components is a tree.

Properties :



$n = 7$

① tree has no cycles

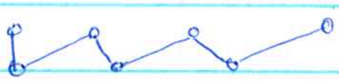
② tree has $(n-1)$ edges

③ tree is a connected graph   no of edges = 6

④ every edge is a bridge. ↦ an edge when removed, the graph gets disconnected.

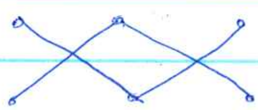
⑤ Any two vertices of tree ~~G~~ are connected by exactly one edge path.

ⓧ q) Which of the following are trees and why ?

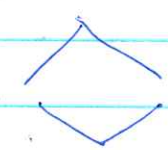


ans : it is a tree because :

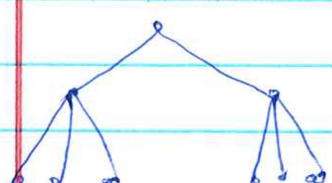
- it is a connected graph
- it has no cycle.



disconnected ⇐



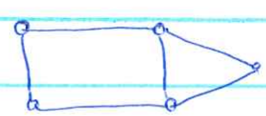
It is not a tree because it is a disconnected graph.



It is a tree because :

it is connected

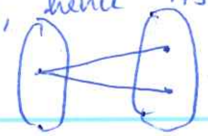
it has no cycle.



It is not a tree because it has a cycle.

Q) Is a tree a bipartite graph ?

Yes. A graph is bipartite iff it contains no cycles of odd length. A tree contains no cycles at all, hence its bipartite



Q) Is $K_{m,n}$ a tree ?

$m = 1, \; n \geq 1$

$n = 1, \; m \geq 1$

Spanning tree :

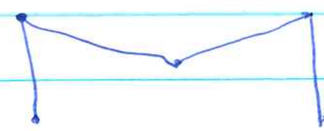
A spanning tree of a graph G, is a subgraph of G which contains of vertices of G and is a tree.

Q) G:

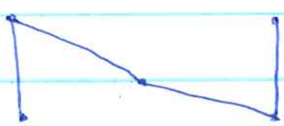
3 spanning trees.

1

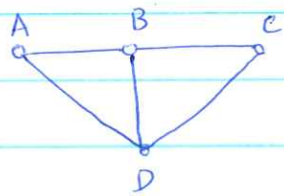
2



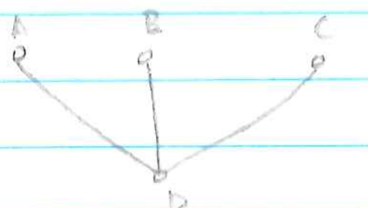
3



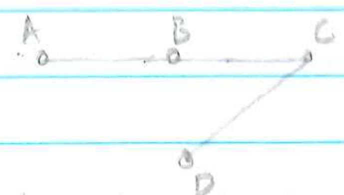
Q) G :



1



2



3